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The Lines of Electric Force Due to a Moving Electron.

By Francis D. Murnaghan.

Introduction.

The values of the electric and magnetic forces in the electromagnetic field due to a moving point charge, or electron, have been given by Liénard,* Wiechert,† Heaviside,‡ and, in detail, by Abraham.§ The expressions are, however, so complicated in the case of perfectly general motion that the form of the lines of force was not obtained except for the simplest case—that of uniform motion in a straight line. The field of force for this case was known || long before that for the general case was worked out, and it was shown that the lines of electric force at any time t were straight lines through the position of the electron at that time t; a result which may be expressed by saying that an electron in uniform motion along a straight line convects its field along with it.

The form of the lines of electric force is of interest in connection with a theory, advanced by Sir J. J. Thomson,¶ of the structure of the ether and the nature of Röntgen rays and light. Following Faraday,** Thomson regards the lines of force not merely as an abstract mathematical concept but as physical realities. He makes, however, one important modification in Faraday's theory. If we have two charges of opposite sign they will be joined by lines of force, the direction of a line at any point being the direction of the electric force at that point. These are the lines of force which Faraday imagined as physical realities endowed with such properties as tension along their length and mutual transverse repulsion. In Thomson's modification of the theory each charge is supposed to carry its own lines of force with it independently of the presence of other charges. In a field containing many charges there will be many lines

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* *L'éclairage électrique*, Vol. XVI (1898), pp. 5, 53, 106.

† *Arch. néerlandaises* (2), Vol. V (1900), p. 549.

‡ *Nature*, Vol. LXVII (1902), p. 6.

§ *Ann. d. Phys.*, Vol. XIV (1904), p. 236; "Theorie der Elektrizität," Band 2, §§ 13–15, Leipzig (1914).

|| O. Heaviside, *Phil. Mag.*, Vol. XXVII (1889), p. 324.

¶ Thomson, *Proc. Camb. Phil. Soc.*, Vol. XV (1909), p. 65.

** A full account of Faraday's theory is given by Thomson in his "Recent Researches in Electricity and Magnetism," Chapter I.

of force crossing at any point and these give a resultant electric force whose direction is that of the Faraday line at the point. Accordingly, the lines of force attached to a point charge are conceived by Thomson as more fundamental than the idea of the point charge itself and may, in fact, be used to define it.

To account for results obtained in experiments on the ionization of gases by Röntgen rays and ultra-violet light Thomson[*] has modified his theory by supposing that the lines of force are not distributed in a continuous manner round the point charge. (He had previously[†] made this modification in Faraday's concept.)

SECTION 1.

Recently a method was given by Bateman[‡] which reduces the problem of finding the equations, at any time t, of the lines of electric force due to a point charge moving in any way, to the solution of a differential equation of Riccati's type. It will be shown that this equation can always be integrated if the path of the electron lies wholly in a plane. The following cases, which are those of most interest, of motion in a plane have been worked out in detail:

(a) Rectilinear motion with any velocity and acceleration;
(b) Uniform motion in a circle.

It has been found possible to integrate the equation in one case of motion in three dimensions—that of uniform motion in a circular helix. Further, it has been found that if one solution of the Riccatian equation is known a second can be derived from it and accordingly the general solution is reduced to the evaluation of a single quadrature.

It will be convenient to give here a brief account of the method by which Bateman obtained the Riccatian equation referred to. Let C be the curve along which the electron is moving and let $x=\xi$, $y=\eta$, $z=\zeta$ be the coordinates of the point of C, occupied by the electron at time τ, so that ξ, η, ζ are functions of τ. Denote differentiations with regard to τ by primes so that the velocity of the electron is $\overline{v}=(\xi', \eta', \zeta')$ and its acceleration is $\overline{v}'=(\xi'', \eta'', \zeta'')$; where bars are used to distinguish vector quantities from scalars. Let P be any point in space. Then if x, y, z are the coordinates of P, a disturbance emanating from the electron at time τ will reach P at time t if $c(t-\tau)=r$ where c is the velocity of light and $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$. Denote by $\overline{r}$ the vector

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[*] *Phil. Mag.*, Vol. XIX (1910), p. 301.

[†] *Proc. Camb. Phil. Soc.*, Vol. XIV, p. 417. (For this and other theories as to the structure of the ether see Bateman "Electrical and Optical Wave Motion," Camb., 1915, Chapter 8.

[‡] *Bulletin Amer. Math. Soc.*, 2d Series, Vol. XXI (1915), No. 6, p. 308; *American Journal*, April (1915).

OP (where O is the position of the electron at time τ) and let (l, m, n) be the direction cosines of $\bar{r}$. Then the electric force at P at time t is known* to be

$$E=\frac{e}{r^2}\left\{\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right\}\left\{1-\frac{v^2}{c^2}+\frac{(\bar{v}'\bar{r})}{c^2}\right\}\left(\frac{\partial\tau}{\partial t}\right)^3-\frac{e\,\bar{v}'}{r\,c^2}\left(\frac{\partial\tau}{\partial t}\right)^2,$$

where e is the charge on the electron, round brackets denoting scalar products.

If we differentiate the equation $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$ we get $\frac{\partial r}{\partial\tau}=\lambda-c$ where $\lambda=c-\frac{1}{r}(\bar{r}\bar{v})$ and since $c(t-\tau)=r$ we have $c\frac{\partial t}{\partial\tau}=\lambda$. We shall be interested in the direction only of E at P and an easy transformation of the expression for E shows that it has the direction of the vector

$$(c^2-v^2)\left\{\bar{v}-c\frac{\bar{r}}{r}\right\}+r\left\{\lambda\bar{v}'+\left(\bar{v}-c\frac{\bar{r}}{r}\right)p\right\},$$

where $p=l\xi''+m\eta''+n\zeta''=\frac{1}{r}(\bar{r}\bar{v}')$.

Let P' be a point consecutive to P and let its coordinates be $x+dx$, $y+dy$, $z+dz$. P' will be reached at time t by a disturbance which has emanated from the electron at time $\tau+d\tau$; at this time let O' be the position of the electron so that $\xi+d\xi$, $\eta+d\eta$, $\zeta+d\zeta$ are the coordinates of O'. The x component of the vector equation $\overline{PP'}=\overline{PO}+\overline{OO'}+\overline{O'P'}$ may be written

$$dx=-rl+d\xi+c(l+dl)(t-\tau-d\tau) \quad\text{or}\quad \frac{dx}{d\tau}=r\frac{dl}{d\tau}+\frac{d\xi}{d\tau}-cl.$$

Thus, if $\bar{s}$ is the vector $\overline{PP'}$ we have $\bar{s}'=\left(\bar{v}-c\frac{\bar{r}}{r}\right)+r(l', m', n')$ where, as previously, (l', m', n') is used to denote the vector whose components are l', m', n'. Comparing this with the expression giving the direction of the electric force at P, we see that PP' is the direction of the electric force at P if the three equations

$$(c^2-v^2)\frac{d}{d\tau}(l, m, n)=\lambda(\xi'', \eta'', \zeta'')+p(\xi'-cl, \eta'-cm, \zeta'-cn)\;\ldots\ldots \quad (A)$$

are satisfied. From the manner in which these equations were derived we see that, if (l, m, n) is a solution of these equations satisfying also the condition $l^2+m^2+n^2=1$; the equations

$$x=\xi+c(t-\tau)l,\quad y=\eta+c(t-\tau)m,\quad z=\zeta+c(t-\tau)n$$

(where τ is a parameter satisfying the inequality $\tau<t$) give a line of electric force at time t. It will be noticed that, if at each position of the electron we imagine a particle projected with velocity c in the direction (l, m, n), the

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* Abraham, "Theorie der Elektrizität."

aggregate of these particles will at any time t form a line of electric force. All the lines of force at any time t pass through the position of the electron at that time.

In order to obtain the solutions (l, m, n) of (A) which satisfy $l^2+m^2+n^2=1$ put $\sigma=\frac{l+im}{1+n}=\frac{1-n}{l-im}$ where $i^2=-1$. After an easy reduction we obtain the Riccatian equation

$$2(c^2-v^2)\frac{d\sigma}{d\tau}=\phi''(c-\zeta')+\phi'\zeta''-2\sigma\{c\zeta''+i(\xi'\eta''-\xi''\eta')\}+\sigma^2\{\psi'\zeta''-(c+\zeta')\psi''\}$$

where $\phi=\xi+i\eta$ and ψ is its conjugate $\xi-i\eta$.

If σ is a solution of this equation and if its conjugate is s, $l=\frac{\sigma+s}{1+\sigma s}$, $m=-i\frac{\sigma-s}{1+\sigma s}$, $n=\frac{1-\sigma s}{1+\sigma s}$ is a solution of the equations (A) which satisfies $l^2+m^2+n^2=1$.

It is interesting to note that the equations (A) are a generalization of the equations

$$\frac{d}{d\tau}(l, m, n)=(mr-nq,\ np-lr,\ lq-mp)\ \ldots. \tag{B}$$

to which they reduce on making $c=0$ and $v^2(p, q, r)=[\bar{v}'\bar{v}]$ the square brackets being used to denote a vector product. These equations occur in the kinematics of a rigid body and have been considered at length by Darboux.[*] Several writers[†] have extended them to the case of four or more variables and have obtained results similar to those found by Darboux.

§ 2. ASSOCIATED DIRECTIONS OF PROJECTION.

A solution (l, m, n) of the equations

$$(c^2-v^2)\frac{d}{d\tau}(l, m, n)=\lambda(\xi'', \eta'', \zeta'')+p(\xi'-cl, \eta'-cm, \zeta'-cn),$$

which have been mentioned in the preceding paragraph, will be called a *direction of projection* provided $l^2+m^2+n^2=1$. It will now be shown that to every direction of projection there corresponds, in an involutory manner, a second direction which shall be called the *associated direction of projection.*

Let O be the position at time τ of the moving electron so that the Cartesian coordinates of O are $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$. Let OP be a direction of projection at O so that (l, m, n), the direction cosines of OP, satisfy the fundamental equations (A) of § 1. Let Q be the position which would have been occupied by the electron at a time $t>\tau$ if it had continued to move with velocity $v=(\xi', \eta', \zeta')$ along the tangent to its path at O. Thus, the coordinates of Q are

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[*] "Théorie des Surfaces," Tome 1, Chapters 2 and 3.

[†] Craig, Cole, Eiesland, *Amer. Journal*, Vol. XX (1898); Hatzidakis, *Amer. Journal*, Vol. XXIII (1901); Eiesland, *Amer. Journal*, Vol. XXVIII (1906); Matsumoto, *Memoirs of Coll. of Science*, Kyoto Imp. Univ., Vol. I, No. 7, January (1916).

$$(x_1, y_1, z_1)=(\xi+\xi'\,\overline{t-\tau},\ \eta+\eta'\,\overline{t-\tau},\ \zeta+\zeta'\,\overline{t-\tau}).$$

If now we make $OP=c(t-\tau)$ and draw a sphere with centre O and radius OP, then PQ will meet the sphere again in a point P'. *OP' is the associated direction of projection at O.* To prove this write $c(t-\tau)=r$ so that the coordinates of P are $(x_0, y_0, z_0)=(\xi+rl,\ \eta+rm,\ \zeta+rn)$ and PQ has the equations $x-x_1/x_1-x_0=y-y_1/y_1-y_0=z-z_1/z_1-z_0=\theta$, say. But $x_1-x_0=(t-\tau)(\xi'-cl)$ and so we have three equations of the type $x-\xi=\xi'(t-\tau)+\theta(\xi'-cl)(t-\tau)$. Let (x, y, z) be the coordinates of P'. P' being on PQ we have $x-\xi=(t-\tau)\{\xi'+\theta_1(\xi'-cl)\}$ where θ_1 is the value of the parameter θ giving P'. Since P' is on the sphere we have the equation

$$(x-\xi)^2+(y-\eta)^2(z-\zeta)^2=c^2(t-\tau)^2$$

and from these two equations we get

$$\theta_1^2\{c^2+v^2-2\frac{c}{r}(\bar{r}\bar{v})\}+2\theta_1\{v^2-\frac{c}{r}(\bar{r}\bar{v})\}+v^2-c^2=0.$$

The roots of this equation are $\theta_1=-1$ and $\theta_1=(c^2-v^2)/\left[c^2+v^2-2\frac{c}{r}(\bar{r}\bar{v})\right]$. The root $\theta_1=-1$ is the parameter of the point P itself. The second root is the parameter of P' and so if (L, M, N) are the direction cosines of OP', so that $x-\xi=c(t-\tau)L$, etc., we obtain the following three equations for L, M, N:

$$(\xi'-cL,\ \eta'-cM,\ \zeta'-cN)=h(v^2-c^2)(\xi'-cl,\ \eta'-cm,\ \zeta'-cn),$$

where $h^{-1}=c^2-2\frac{c}{r}(\bar{r}\bar{v})+v^2$. If as in § 1 we use the abbreviations

$$\lambda=c-\frac{1}{r}(\bar{r}\bar{v}),\ \mu=c^2-v^2,\ \text{then } h^{-1}=2c\lambda-\mu.$$

Write $\bar{R}$ to denote the vector OP', and Γ to denote the expression $c-\frac{1}{r}(\bar{R}\bar{v})$; then from the values obtained for (L, M, N) it is easy to deduce the equation $\Gamma=h\lambda\mu$.

In order to prove that OP' is a direction of projection we have merely to show that (L, M, N) satisfy the equations (A) of § 1. In other words, it must be shown that the three equations

$$\mu\frac{d}{d\tau}(L, M, N)=\Gamma(\xi'', \eta'', \zeta'')+P(\xi'-cL, \eta'-cM, \zeta'-cN)$$

are true, where $P=L\xi''+M\eta''+N\zeta''=\frac{1}{r}(\bar{R}\bar{v}')$ and, as before, $\bar{v}'$ is the vector acceleration of the electron at O. If now we multiply the equations (A) by ξ', η', ζ', respectively, and add they give $\mu(p+q)=\lambda\{cp+(\bar{v}\bar{v}')\}$ where $q=l'\xi'+m'\eta'+n'\zeta'$. This equation is equivalent to

$$-\mu\lambda'=\lambda\{cp+(\bar{v}\bar{v}')\}.$$

Again $\xi'-cL=-\mu h(\xi'-cl)$. Our object is to obtain L' so we differentiate this equation and it gives

$$cL'-\xi''=\mu h(\xi''-cl')-2h(\bar{v}\bar{v}')(\xi'-cl)-2\mu h^2(\xi'-cl)[c\lambda'+(\bar{v}\bar{v}')].$$

On combining the first two terms on the right-hand side of this equation with the term $-\xi''$ on the left-hand side we obtain, after substituting for l' its value from the equation (A) of § 1 and for λ' its value just given

$$cL'=h\{c\lambda\xi''-(\xi'-cl)[cp+2(\bar{v}\bar{v}')]\}+2h^2(\xi'-cl)\{c^2\lambda p+(c\lambda-\mu)(\bar{v}\bar{v}')\}$$

or

$$L'=h\{\lambda\xi'-(\xi'-cl)p\}+2h^2\lambda(\xi'-cl)\{cp-(\bar{v}\bar{v}')\}.$$

Again from the values already given for (L, M, N) we derive

$$cP=(\bar{v}\bar{v}')+\mu h\{(\bar{v}\bar{v}')-cp\} \text{ or } P=h[2\lambda(\bar{v}\bar{v}')-\mu p] \text{ since } h^{-1}=2c\lambda-\mu.$$

Substituting these values of L', P in the equation $\mu L'=\Gamma\xi''+(\xi'-cL)P$ we see on dividing across by $\mu hp(\xi'-cl)$ that this equation is true if $1-2hc\lambda+\mu h=0$ which is so since $h^{-1}=2c\lambda-\mu$.

This proves the theorem stated. If we know a particular solution $OP=(l, m, n)$ of the equations (A) we can obtain a second solution $OP'=(L, M, N)$ in the manner described at the beginning of this paragraph. Darboux* has given a similar result for the equations (B) of § 1.

There is one particular case, when the associated direction coincides with the original direction of projection, in which we can not obtain by this method a new solution of the differential equations. In this case PP' is a tangent line to the sphere and so $l\xi'+m\eta'+n\zeta'=v\frac{c}{v}$ or $\lambda=0$. It is easy, as a matter of fact, to verify that if we put $\lambda=0$ in the equations for L, M, N that $L=l$, $M=m$, $N=n$. In this case the velocity v of the electron is greater than c, the velocity of light. From the equation for λ' we see that when $\lambda=0$ λ' also $=0$. In the cases of motion for which the solution has been worked out it will be seen that there are always a single infinity of solutions compatible with $\lambda=0$.

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* "Théorie des Surfaces," Tome 1 (1887), p. 23.

§ 3. GENERAL SOLUTION WHEN MOTION IS IN ONE PLANE.

Let θ be the angle between OP and OQ so that $v\cos\theta = l\xi' + m\eta' + n\zeta'$. It will now be seen that if the associated directions OP, OP' are distinct and equally inclined to OQ, the motion of the electron is in one plane and the solution can be found. OP and OP' are equally inclined to OQ if $\cos\theta = \dfrac{v}{c}$ for then PP' is perpendicular to OQ. Also since $\lambda = c - v\cos\theta$ we have $\mu = c\lambda$, and so $-c\lambda' = 2(\bar{v}\bar{v}')$. On substitution in the equation $-\mu\lambda' = \lambda\{cp + (\bar{v}\bar{v}')\}$ of § 2 we get $-c\lambda' = cp + (\bar{v}\bar{v}')$ or $cp = (\bar{v}\bar{v}')$. Substituting in the first of the equations (A) this gives $\mu(\xi'' - cl') = -(\xi' - cl)(\bar{v}\bar{v}')$ which is at once integrable, giving $\xi' - cl = a(c^2 - v^2)^{\frac{1}{2}}$ where a is a constant of integration. Similarly, we obtain the equations $\eta' - cm = b(c^2 - v^2)^{\frac{1}{2}}$, $\zeta' - cn = d(c^2 - v^2)^{\frac{1}{2}}$ where b and d are constants of integration. On multiplication by ξ', η', ζ', respectively, and addition, these three integrals give $a\xi' + b\eta' + d\zeta' = 0$ (remembering $v^2 = c^2 - c\lambda$).

Similarly, on using ξ'', η'', ζ'' as multipliers, we get, since $(\bar{v}\bar{v}') = cp$,

$$a\xi'' + b\eta'' + d\zeta'' = 0.$$

Thus, the velocity and the acceleration of the electron are in the same plane; hence, if a solution of the fundamental equations can be found which satisfies the equation $c\cos\theta = v$, the motion of the electron is in one plane. Conversely, if the motion is plane it is easy to pick out a solution for which $c\cos\theta = v$. The theorem of associated directions, then, gives a second solution which also satisfies $c\cos\theta = v$. Knowing these two solutions the integration of the Riccatian equation of § 1 is effected by means of a single quadrature.

Take the plane of motion to be $z = 0$ so that $\zeta' = 0$, $\zeta'' = 0$. It is evident that $cl = \xi'$, $cm = \eta'$ satisfy the first two equations. These values of l, m satisfy $c\cos\theta = v$. To determine n we have the condition $l^2 + m^2 + n^2 = 1$, giving $cn = \pm(c^2 - v^2)^{\frac{1}{2}}$. Either of these values of n satisfies the third equation which is, in this case, $\mu n' = -cn(l\xi'' + m\eta'')$. Hence, the solutions which give $c\cos\theta = v$ are $cl = \xi'$, $cm = \eta'$, $cn = \pm\mu^{\frac{1}{2}}$.

To obtain the general direction of projection it is necessary to solve the Riccatian equation. In the case of plane motion this is somewhat simplified and is $2(c^2 - v^2)\dfrac{d\sigma}{d\tau} = c\phi'' - 2i\sigma(\xi'\eta'' - \xi''\eta') - c\sigma^2\psi''$ where $\sigma(1 + n) = l + im$. Substituting for l, m, n the values just given we have the two particular solutions σ_1, σ_2 given by $c(1 + \omega)\sigma_1 = \phi'$, $c(1 - \omega)\sigma_2 = \phi'$ where $c\omega = (c^2 - v^2)^{\frac{1}{2}}$. It is easy to verify that σ_1, σ_2 satisfy the Riccatian equation.

The general solution of the Riccatian equation is, therefore,*

$$\sigma=\sigma_2+\frac{\sigma_1-\sigma_2}{1+Ae^{\int(\sigma_1-\sigma_2)R\,d\tau}},$$

where A is a constant of integration and $2(c^2-v^2)R=-c\psi''$. A is, in general, complex since σ is complex.

Thus, when the motion is plane, the problem is reduced to the integration

$$\int(\sigma_1-\sigma_2)R\,d\tau \text{ or } c\int\frac{(\xi''-i\eta'')\,d\tau}{(\xi'-i\eta')\sqrt{c^2-\xi'^2-\eta'^2}}.$$

The integration can always be effected if the scalar quantity v is constant, and in this case the integral is $[c/(c^2-v^2)^{\frac{1}{2}}]\log\psi'$, so that

$$\sigma=\sigma_2+\frac{\sigma_1-\sigma_2}{1+A\psi'^{\frac{1}{\omega}}}.$$

If $\eta=0$, $\eta'=0$, $\eta''=0$, it is evident that the integration can be carried out. This is the case of rectilinear motion which will be treated in detail. The remaining case of most interest is that of uniform motion in a circle and the solution is in this case given in terms of the trigonometric functions. In general, the explicit determination of the equations of the lines of force depends on the evaluation of the integral given above.

§ 4. PARTICULAR CASES OF PLANE MOTION.

(a) *Rectilinear Motion.*

Taking the x-axis as the line of motion we see from § 3 that the solution depends on the integration of $c\int\frac{\xi''\,d\tau}{\xi'\sqrt{c^2-\xi'^2}}$. In this particular case, however, the general solution can be obtained at once directly from the Riccatian equation which becomes $2(c^2-\xi'^2)\sigma'=c\xi''(1-\sigma^2)$.

This gives $\log\frac{1+\sigma}{1-\sigma}=\frac{1}{2}\log\frac{c+v}{c-v}+\log A$ (for $\xi'=v$). Write $A=\alpha+i\beta$ and we have $\frac{1+\sigma}{1-\sigma}=(\alpha+i\beta)\left(\frac{c+v}{c-v}\right)^{\frac{1}{2}}$. Whence,

$$\sigma=\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}-(c-v)\}\div\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}+(c-v)\}.$$

From this point we must distinguish between the cases $v\lessgtr c$.

Linear Motion When $v<c$.

$(c^2-v^2)^{\frac{1}{2}}$ is real and so if s denotes the conjugate of σ we have

$$s\{(\alpha-i\beta)(c^2-v^2)^{\frac{1}{2}}+(c-v)\}=(\alpha-i\beta)(c^2-v^2)^{\frac{1}{2}}-(c-v).$$

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* Cohen, "Differential Equations," p. 175.

Forming the product σs we find

$$\sigma s\{(\alpha^2+\beta^2)(c^2-v^2)+(c-v)^2+2(c-v)(c^2-v^2)^{\frac{1}{2}}\alpha\}$$
$$=(\alpha^2+\beta^2)(c^2-v^2)+(c-v)^2-2(c-v)(c^2-v^2)^{\frac{1}{2}}\alpha,$$

and since $(1+\sigma s)l=\sigma+s$, $(1+\sigma s)im=\sigma-s$, $(1+\sigma s)n=1-\sigma s$ we have

$$Rl=(c+v)(\alpha^2+\beta^2)-(c-v),\quad Rm=2\beta(c^2-v^2)^{\frac{1}{2}},\quad Rn=2\alpha(c^2-v^2)^{\frac{1}{2}},$$

where $R=(c+v)(\alpha^2+\beta^2)+(c-v)$.

The lines of force at time t are, with these values for l, m, n,

$$x=\xi+c(t-\tau)l,\quad y=c(t-\tau)m,\quad z=c(t-\tau)n,$$

and α, β are two arbitrary real constants.

It is seen that $\alpha y=\beta z$ or the lines of force lie in planes through the direction of motion a result evident, à priori, from symmetry. If v is constant, so also are l, m, n, and we get the well-known result* that in the case of uniform motion in a right line the line of electric force at time t through any point x, y, z is got by joining x, y, z to the position of the electron at that time t. Since the lines of force are symmetrical round the direction of motion it is sufficient to consider the lines in the plane $z=0$, so that $\alpha=0$. For instance, in the case of simple harmonic motion we may write $\xi=a\cos p\tau$ and the lines of electric force in the xy-plane are $x=a\cos p\tau+c(t-\tau)l$, $y=c(t-\tau)m$, where

$$l\{\beta^2(c-ap\sin p\tau)+(c+ap\sin p\tau)\}=\beta^2(c-ap\sin p\tau)-(c+ap\sin p\tau),$$
$$m\{\beta^2(c-ap\sin p\tau)+(c+ap\sin p\tau)\}=2\beta(c^2-a^2p^2\sin^2 p\tau)^{\frac{1}{2}}.$$

The single arbitrary constant β gives the single infinity of lines in the plane. To get an idea of the shape of the lines take the particular line $\beta=1$. Then, $cl=-ap\sin p\tau$, $cm=(c^2-a^2p^2\sin^2 p\tau)^{\frac{1}{2}}$; put $t=0$ and $\tau=-\tau'$ so that τ' takes positive values. Then, $x=a\cos p\tau'+c\tau' l$, $y=c\tau' m$, where $cl=ap\sin p\tau'$. Thus, $x=a(\cos p\tau'+p\tau'\sin p\tau')$, $y=\tau'(c^2-a^2p^2\sin^2 p\tau')^{\frac{1}{2}}$, where τ' takes positive values. If a is small we have at distances far from the origin

$$x^2+y^2=c^2\tau'^2,\quad x=a(\cos p\tau'+p\tau'\sin p\tau').$$

The line of force is of an oscillatory character with increasing amplitude about the line $x=0$.

Case of Uniformly Accelerated Motion in a Straight Line.

Let g be the uniform acceleration and suppose the electron to start from rest at the origin at time $\tau=0$. Then, $\xi=\frac{1}{2}g\tau^2$, $v=g\tau$ so that

$$Rl=\beta^2(c+g\tau)-(c-g\tau),\quad Rm=2\beta(c^2-g^2\tau^2)^{\frac{1}{2}},\ \text{where}\ R=\beta^2(c+g\tau)+(c-g\tau).$$

Consider the particular line $\beta=1$. Then, $R=2c$ and $cl=g\tau$ and the equations to the line of force are $x=\frac{1}{2}g\tau^2+(t-\tau)g\tau$, $y=(t-\tau)(c^2-g^2\tau^2)^{\frac{1}{2}}$.

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* J. J. Thomson, *Phil. Mag.*, Vol. XI (1881), p. 229; O. Heaviside, *Phil. Mag.*, Vol. XXVII (1889), p. 324.

These equations take a somewhat simpler form if we transfer the origin to the position of the electron at time t. Then, $x=-\frac{1}{2}g(t-\tau)^2$ and taking $t-\tau$ as the parameter s we have $x=-\frac{1}{2}gs^2$, $y^2=s^2\{c-g(t-s)\}\{c+g(t-s)\}$, where s takes values from t to zero. On eliminating s this is a quartic curve

$$\left\{y^2+4x^2+\frac{2x}{g}(c^2-g^2t^2)\right\}^2+8gt^2x^3=0.$$

It is easy to see how the lines of force due to a point charge in rectilinear motion are altered in shape when the moving charge is accelerated or retarded. For example, imagine the charge to have been moving with uniform velocity for a long time and, then, to be suddenly affected with a retardation g, and consider, as above, the line $\beta=1$ for which $l=\frac{v}{c}$; the electron, being at $\tau=0$ at $x=0$, is reduced to rest at time $\tau=\frac{v}{g}$. Consider the line of force at this time. The part given by the aggregate of particles imagined projected at times $\tau<0$ is a straight line which would pass if produced through the point $x=\frac{v^2}{g}$. At the point where this straight line meets the line $cl=v$ through the origin, the line of force bends and passes through the point $x=\frac{v^2}{2g}$ in a direction perpendicular to the line of motion. At any later time t the line of force is made up of three parts: (1) A straight line part which goes, if produced, through $x=vt$, (2) a bent part beginning where this line meets the sphere $x^2+y^2+z^2=c^2t^2$ and ending at $x=\frac{v^2}{2g}$, $y=c\left(t-\frac{v}{g}\right)$, and (3) the part of the line $x=\frac{v^2}{2g}$ from this point to $y=0$. A similar result holds for the lines got from other values of β. It is evident that if the electron is stopped instantaneously we can get the lines of force in the following way: Draw the pencil of lines through $x=vt$, $y=0$. The sphere with centre at origin and radius ct cuts these lines in points where the lines of force bend suddenly, the remaining portions being the radii of this sphere. This explains how the discontinuity due to the sudden stoppage of an electron spreads out in a spherical wave. The commonly accepted theory of Röntgen rays is that they are pulses of this type due to the sudden stoppage of cathode particles. It is important to notice that in all cases the direction of projection is determined by the velocity at the point of projection and is independent of its acceleration.

If we give particular values to v, g, c, it is easy to draw the bent portion of the line of force due to the retarded part of the motion. For instance, take $v=4$, $g=c=8$; then the electron is reduced to rest in time $\tau=\frac{1}{2}$, its position at that time being $x=1$. Taking $\beta=1$ we have $l=\frac{v}{c}$. First take $t=\frac{1}{2}$, so that we are drawing the line for the instant when the electron is reduced to rest. Then, if $0<\tau<\frac{1}{2}$, $l=\frac{4-8\tau}{8}=\frac{1}{2}-\tau$, $\xi=4(\tau-\tau^2)$. The distance from ξ to the corresponding point on the line of force is $c(t-\tau)=4(1-2\tau)=8l$. It is found that the bent portion of the line of force resembles closely the part in the positive quadrant of an ellipse whose axes are in the ratio of 10 : 34. Having drawn the line of force at time $t=\frac{1}{2}$ we get the form of the same line at any subsequent time by producing the radii vectores from ξ to the corresponding point on the line of force at time $t=\frac{1}{2}$, a constant distance. This method of deducing the successive positions of a given line of force from the position at any one time follows from the equations of the lines of force and is applicable in the general case.

Motion in a Straight Line When $v>c$.

In the equation

$$\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}+(c-v)\}\sigma=\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}-(c-v)\}$$

$(c^2-v^2)^{\frac{1}{2}}$ is now a pure imaginary. Writing it in form $i(v^2-c^2)^{\frac{1}{2}}$ we see that s, the conjugate of σ, is given by the equation

$$\{(\alpha-i\beta)(v^2-c^2)^{\frac{1}{2}}+i(c-v)\}s=(\alpha-i\beta)(v^2-c^2)^{\frac{1}{2}}-i(c-v).$$

From these we obtain as before

$$Rl=(\alpha^2+\beta^2)(v+c)-(v-c),\quad Rm=2\alpha(v^2-c^2)^{\frac{1}{2}},\quad Rn=2\beta(v^2-c^2),$$

where $R=(\alpha^2+\beta^2)(v+c)+(v-c)$.

The lines of force lie in planes through the axis of x; putting $\beta=0$ we obtain the lines in the plane xy. It will be noticed that, for these, if $\alpha=1$, $l=\frac{c}{v}$, and so $\lambda=0$. Hence, in the case of rectilinear motion with constant velocity $v>c$ the generators of the cone $l=\frac{c}{v}$ are directions of projection. The cone with vertex at $x=\xi(t)$ and semi-vertical angle $\alpha=\sin^{-1}\frac{c}{v}$ accordingly separates the regions where there are lines of electric force from those regions, where there are no lines of force. The generators of the cone are lines of electric force. This is a known result.* If v is not constant, these successive cones envelope a surface of revolution, the meridians on which are lines of electric force, and which enclose all the lines of force.

Motion in a Circle with Uniform Velocity.

Let p be the constant angular velocity and d the radius of the circle. Then, the plane of the circle being $z=0$, and the centre of the circle being the origin of coordinates, we have $\xi=a\cos p\tau$, $\eta=a\sin p\tau$, $\zeta=0$; whence, $\phi=ae^{ip\tau}$ and the particular solutions of the Riccatian equation are σ_1, σ_2, where

$$c(1+\omega)\sigma_1=iape^{ip\tau},\ c(1-\omega)=iape^{ip\tau} \text{ and } c^2\omega^2=c^2-v^2=c^2-a^2p^2.$$

On substituting these values in the general solution for plane motion it is found that

$$\sigma=ie^{ip\tau}\left[\frac{c(1+\omega)}{ap}+\frac{1}{(\alpha+i\beta)e^{\frac{-ip\tau}{\omega}}-ap/2c\omega}\right].\dagger$$

If $v<c$, ω is real; if $v>c$, ω is a pure imaginary, so there will be two distinct solutions according as $v \gtrless c$.

(a) *Case When $v<c$.*

Since ω is real, s is got by merely changing the sign of i wherever it appears in the expression for σ. Adding we have

$$\sigma+s=-\frac{2c(1+\omega)}{ap}\sin p\tau-2\frac{P\sin p\tau+Q\cos p\tau}{R},$$

where P, Q, R are defined by the equations

$$P=\alpha\cos p\tau/\omega+\beta\sin p\tau/\omega-ap/2c\omega,$$
$$Q=\alpha\sin p\tau/\omega-\beta\cos p\tau/\omega,$$
$$R=(\alpha^2+\beta^2)+a^2p^2/4c^2\omega^2-ap/c\omega\,(\alpha\cos p\tau/\omega+\beta\sin p\tau/\omega).$$

Similarly, on subtracting we find

$$-i(\sigma-s)=2c(1+\omega)/ap\cos p\tau+2\frac{P\cos p\tau-Q\sin p\tau}{R},$$

and on multiplication

$$\sigma s=\frac{c^2(1+\omega)^2}{a^2p^2}+\frac{\frac{2c(1+\omega)}{ap}P+1}{R};$$

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* O. Heaviside, "Electrical Papers."

† The integral $\int\frac{\psi''}{\psi'\omega}d\tau$ of § 3 is, in this case, $\frac{1}{\omega}\log\psi'$ and $\psi=ae^{-ip\tau}$.

from these equations we obtain $R(1+\sigma s)=S$, $R(1-\sigma s)=T$, where S and T are defined by the equations

$$S=(\alpha^2+\beta^2)\frac{2}{1-\omega}+\frac{1-\omega}{2\omega^2}-\frac{2ap}{c\omega}\left(\alpha\cos\frac{p\tau}{\omega}+\beta\sin\frac{p\tau}{\omega}\right),$$

$$T=(\alpha^2+\beta^2)\frac{2\omega}{\omega-1}+\frac{1-\omega}{2\omega},$$

and hence l, m, n are given by

$$l\equiv\frac{\sigma+s}{1+\sigma s}=-\frac{ap}{c}\sin p\tau+\frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\sin p\tau-Q\cos p\tau\right\},$$

$$m\equiv-i\frac{\sigma-s}{1+\sigma s}=\frac{ap}{c}\cos p\tau-\frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\cos p\tau+Q\sin p\tau\right\},$$

$$n=\frac{1-\sigma s}{1+\sigma s}=\frac{T}{S},$$

and with these values of l, m, n the parametric equations to the lines of force are $x=a\cos p\tau+c(t-\tau)l$, $y=a\sin p\tau+c(t-\tau)m$, $z=c(t-\tau)n$.

By a suitable choice of the constants α, β so as to get particular lines of force these expressions may be considerably simplified. Thus, if we put $\alpha=0$, $\beta=0$, we have $l=-(ap/c)\sin p\tau$, $m=(ap/c)\cos p\tau$, $n=\omega$, and the line of force is given by the equations

$$x=a\cos p\tau-ap(t-\tau)\sin p\tau,\quad y=a\sin p\tau+ap(t-\tau)\cos p\tau,\quad z=c\omega(t-\tau).$$

It is obvious that it lies on the hyperboloid of revolution

$$(x^2+y^2)/a^2-p^2z^2/c^2\omega^2=1$$

which contains the circle of motion. Writing this equation in the form

$$(x/a-pz/c\omega)(x/a+pz/c\omega)=(1+y/a)(1-y/a),$$

we see that the directions of projection are generators of the hyperboloid. The form of the line of force is evident if we consider its projection on the plane of motion. The equations of the projection are

$$x=a\cos p\tau-ap(t-\tau)\sin p\tau,\quad y=a\sin p\tau+ap(t-\tau)\cos p\tau,\quad z=0;$$

giving

$$(x-a\cos p\tau)+(y-a\sin p\tau)\tan p\tau=0$$

$$\text{and }(x-a\cos p\tau)^2+(y-a\sin p\tau)^2=a^2p^2(t-\tau)^2.$$

These are the equations to an involute of the circle $x^2+y^2=a^2$; the involute meeting the circle at the position of the electron at time $\tau=t$. To obtain, then, a line of electric force due to an electron moving with velocity $v<c$ in a circle we unwrap a string which is wound round the circle, the tracing point at the end of the string leaving the circle at the position of the electron at time t. Having obtained this involute we project it by lines parallel to the axis of z on the hyperboloid $(x^2+y^2)/a^2-p^2z^2/c^2\omega^2=1$.

Lines of Force in the Plane of Motion.

To obtain these put $n=0$. Then, $T=0$ or $(\alpha^2+\beta^2)=(\overline{1-\omega}/2\omega)^2$, and so we may write $2\omega\alpha=(1-\omega)\cos\theta$, $2\omega\beta=(1-\omega)\sin\theta$ and these give $\omega^2 S=(1-\omega)k$ where $k=1-(ap/c)\cos(p\tau/\omega-\theta)$, and, finally,

$$kl=\sin p\tau\cos(p\tau/\omega-\theta)-(ap/c)\sin p\tau-\omega\cos p\tau\sin(p\tau/\omega-\theta),$$
$$km=-\cos p\tau\cos(p\tau/\omega-\theta)+(ap/c)\cos p\tau-\omega\sin p\tau\sin(p\tau/\omega-\theta).$$

The single constant θ gives the single infinity of lines in the plane of motion; from symmetry we need only consider a particular value of θ, say $\theta=0$; then,

$$kl=\sin p\tau\cos p\tau/\omega-(ap/c)\sin p\tau-\omega\cos p\tau\sin p\tau/\omega,$$
$$km=-\cos p\tau\cos p\tau/\omega+(ap/c)\cos p\tau-\omega\sin p\tau\sin p\tau/\omega,$$

where the Doppler factor k is now $1-(ap/c)\cos p\tau/\omega$.

Circular Motion When $v>c$.

ω is now a pure imaginary and so it may be written $=i\theta$ where θ is real. The functions σ and s are given by the equations

$$\sigma=\quad ie^{ip\tau}\left\{\frac{c(1+i\theta)}{ap}+\frac{1}{(\alpha+i\beta)e^{-p\tau/\theta}+iap/2c\theta}\right\},$$
$$s=-ie^{-ip\tau}\left\{\frac{c(1-i\theta)}{ap}+\frac{1}{(\alpha-i\beta)e^{-p\tau/\theta}-iap/2c\theta}\right\},$$

whence

$$D(\sigma+s)=-D\left(\frac{2c}{ap}\sin p\tau+\frac{2c\theta}{ap}\cos p\tau\right)+2e^{-p\tau/\theta}\{\beta\cos p\tau-\alpha\sin p\tau\}+\frac{ap}{c\theta}\cos p\tau,$$

where

$$D=\alpha^2e^{-2p\tau/\theta}+\left\{\beta e^{-p\tau/\theta}+\frac{ap}{2c\theta}\right\}^2.$$

Similarly,

$$-iD(\sigma-s)=\frac{2c}{ap}D(\cos p\tau-\theta\sin p\tau)+2e^{-p\tau/\theta}\{\alpha\cos p\tau+\beta\sin p\tau\}+\frac{ap}{c\theta}\sin p\tau$$

$$\text{and }\ \sigma s=1+\frac{2c}{apD}e^{-p\tau/\theta}(\alpha-\theta\beta).$$

Using these values we obtain, after some reduction, and on writing

$$F=(\alpha^2+\beta^2)e^{-2p\tau/\theta}-a^2p^2/4c^2\theta^2,$$
$$G=(\alpha^2+\beta^2)e^{-2p\tau/\theta}+a^2p^2/4c^2\theta^2+\frac{c}{ap}e^{-p\tau/\theta}(\alpha+\beta/\theta),$$

the equations

$$l=-\frac{c}{ap}\left[\sin p\tau+\frac{\theta\{F\cos p\tau+\frac{c}{ap}e^{-p\tau/\theta}(\theta\alpha+\beta)\sin p\tau\}}{G}\right],$$

$$m=\quad\frac{c}{ap}\left[\cos p\tau-\frac{\theta\{F\sin p\tau-\frac{c}{ap}e^{-p\tau/\theta}(\theta\alpha+\beta)\cos p\tau\}}{G}\right],$$

$$n=\quad\frac{c}{ap}e^{-p\tau/\theta}\frac{(\theta\beta-\alpha)}{G}.$$

The lines in the plane of motion are obtained by writing $\alpha = \theta\beta$. Further, $\lambda=0$ reduces to $\theta\alpha+\beta=0$ and with this relation between the constants we find

$$l = \left\{\frac{c\theta}{ap}\cos p\tau\right\}\frac{1-x}{1+x} - \frac{c}{ap}\sin p\tau,$$

$$m = \left\{\frac{c\theta}{ap}\sin p\tau\right\}\frac{1-x}{1+x} + \frac{c}{ap}\sin p\tau,$$

$$n = -\frac{4c}{ap}\,\frac{\alpha\theta^2 e^{-p\tau/\theta}}{1+x},$$

where $x=4\alpha^2\theta^2 e^{-2p\tau/\theta}$. Thus there is a single infinity of lines of electric force whose directions of projection satisfy $\lambda=0$. Putting $\alpha=0$ we get that one of these which lies in the plane of motion and from symmetry this line is the same as all the lines in the plane of motion. Since for this line

$$l = -\frac{c}{ap}\left[\sin p\tau-\theta\cos p\tau\right],\quad m = \frac{c}{ap}\left[\cos p\tau+\theta\sin p\tau\right],\quad n = 0,$$

we have on writing $\sin\phi=\frac{c}{ap}$, $\cos\phi=\frac{c\theta}{ap}$ that $l=\cos(p\tau+\phi)$, $m=\sin(p\tau+\phi)$, and the equations to the line of force are

$$x=a\cos p\tau+c(t-\tau)\cos(p\tau+\phi),$$

$$y=a\sin p\tau+c(t-\tau)\sin(p\tau+\phi).$$

The direction of projection makes a fixed angle with the radius of the circle to the point of projection; the line, therefore, winds round the circle in a manner very similar to its involutes.

Motion in a Helix with Uniform Velocity.

It has been found possible to solve the problem in one case where the motion is not in one plane. If the electron moves in a circular helix with constant velocity so that $\xi=a\cos p\tau$, $\eta=a\sin p\tau$, $\zeta=d\tau$, where d is a constant, the Riccatian equation of § 1 may be written

$$2(c^2-a^2p^2-d^2)\frac{d\sigma}{d\tau} = ap^2(d-c)e^{ip\tau}-2i\sigma a^2p^3+ap^2\sigma^2e^{-ip\tau}(c+d).$$

Making use of the trial solution $Ae^{ip\tau}$ we find that σ_1, σ_2 are particular solutions where

$(c+d)(1+\omega)\sigma_1=iape^{+ip\tau}$, $(c+d)(1-\omega)\sigma_2=iape^{ip\tau}$ and $(c^2-d^2)\omega^2=c^2-d^2-a^2p^2$.

Proceeding in the same way as before the general solution is found to be

$$\sigma = ie^{ip\tau}\left\{\frac{(c-d)}{ap}(1+\omega)+\frac{1}{(\alpha+i\beta)e^{-ip\tau/\omega}-ap/2(c-d)\omega}\right\}$$

and from this on the work is the same as in the case of circular motion. In the case when $c>d$ and $c^2<d^2+a^2p^2$ (so that ω is a pure imaginary) it is found as before that if we write $\omega=i\theta$ that $\lambda=0$ if $\alpha\theta+\beta=0$. Thus, again, there exists a single infinity of lines of force for which the directions of projections satisfy $\lambda=0$. It has not been thought necessary to give the detailed results here as they are not essentially different from those already developed for the case of circular motion.

SECTION 5.

It was thought that it might be possible to find several curves with the same directions of projection. In other words, that, knowing a solution (l, m, n) of the equations (A) of § 1 for a definite curve of motion, it might be possible to find another curve or system of curves crossing the directions of projection for the original curve and such that these lines are also directions of projection for them. It was found that this is not in general the case and if a second path could be obtained in this way the three equations of the type

$$\left[p-\frac{2\lambda\{(\bar{v}\,\bar{v}')+cp\}}{c^2-v^2}\right]\frac{\partial l}{\partial\tau}=(\xi'-cl)\,q+\lambda\frac{\partial^2 l}{\partial\tau^2}$$

would be satisfied. From these it is easy to deduce that

$$\left(\frac{\partial l}{\partial\tau}\right)^2+\left(\frac{\partial m}{\partial\tau}\right)^2+\left(\frac{\partial n}{\partial\tau}\right)^2=a\lambda^2$$

where a is a constant, an equation which is true in the case of rectilinear motion with uniform velocity.

CONCLUSION.

The main results of this paper may be restated as follows: The problem of finding the equations of the lines of electric force due to a moving electron, considered as a point charge, has been reduced to the calculation of an indefinite integral in the case when the motion is all in one plane and so may be regarded as solved. The more important cases of plane motion such as rectilinear motion and motion with constant velocity in a circle have been worked in detail and several lines have been found to have a more or less simple geometrical characterization; in particular, may be mentioned the line for the case of uniform circular motion which lies on a hyperboloid of revolution through the circle of motion and whose projection on the plane of this circle is an involute of the circle. The solution of the problem for the case of uniform motion in a circular helix has been indicated and this is interesting in so far that no general method has been found for cases where the motion of the point charge is not in one plane.